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# Samuelson's multiplier–accelerator model revisited

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As demonstrated by Samuelson, the interplay between the multiplier analysis and the principle of acceleration may generate temporary business cycles. We extend Samuelson's seminal framework in the sense that investors now apply a nonlinear mix of extrapolative and regressive expectation formation rules to predict changes in national income. As it turns out, the emergence of booms and slumps may depend on the sentiment of the investors. If they are optimistic (pessimistic), the economy performs well (badly). Moreover, the model produces sustained and intricate fluctuations in economic activity for realistic values of the marginal propensity to consume and the capital to output ratio.

## I. Introduction

With the advent of Samuelson's (1939) multiplier–accelerator model, modern business cycle theory was born. Samuelson combined the newly arrived Keynesian multiplier analysis with the older principle of acceleration. According to the multiplier analysis, long-run equilibrium output is proportional to autonomous expenditure. The accelerator principle states that induced investment is driven by changes in consumption, respectively by changes in national income. Samuelson demonstrates that the interplay between these two mechanisms may, in principle, yield fluctuations in economic activity. For example, an increase in governmental expenditure leads to an increase in national income, which in turn raises investment. As a result, national income may further increase.

Many more interesting contributions to business cycle modelling have been presented during recent decades, using more sophisticated mechanisms and involving aspects such as the monetary sector or inventory adjustments (Medio, 1992; Day and Chen, 1993; Day, 1999; Rosser, 2000). Although the core elements of Samuelson's model are still valid

(Gandolfo, 1985; Puu, 1989; Hommes, 1995; Puu *et al.*, 2004), it is frequently said that the model has two severe shortcomings. First, it is not able to produce lasting business cycles. Second, empirically observed values of its parameters imply that the trajectory of national income is unstable. Moreover, we would like to add that the model neglects expectations.

The objective of this paper is to amend these drawbacks. We seek to achieve this by allowing investors to rely on a nonlinear mix of extrapolative and regressive expectation formation rules to predict national income. Our one-dimensional nonlinear discrete-time model suggests that economic activity endogenously depends on the mood of the agents. If they are optimistic (pessimistic), output tends to be above (below) its long-run equilibrium value. The model also produces complex (chaotic) business cycles for realistic values of the marginal propensity to consume and the capital to output ratio.

The rest of our paper is organized as follows. Section II repeats Samuelson's business cycle model. In Section III, we introduce our modifications and in Section IV, we present some results. The final section concludes the paper.

## II. The Multiplier–Accelerator Model

Let us briefly recall the seminal business cycle model of Samuelson (1939). National income at time  $t$ ,  $Y_t$ , may be written as the sum of three components: consumption,  $C_t$ , induced private investment,  $I_t$ , and governmental expenditure,  $G_t$ . Therefore,

$$Y_t = C_t + I_t + G_t \quad (1)$$

Consistent with the Keynesian multiplier analysis, the agents consume a constant fraction of their past income

$$C_t = aY_{t-1} \quad (2)$$

where  $0 < a < 1$  stands for the marginal propensity to consume. Induced private investment is proportional to changes in consumption and thus also to changes in national income

$$I_t = b(C_t - C_{t-1}) = ab(Y_{t-1} - Y_{t-2}) \quad (3)$$

As is well known,  $ab > 0$  denotes the capital to output ratio, i.e. the relation between the capital stock and output is fixed. Since governmental expenditure is constant

$$G_t = \bar{G} \quad (4)$$

national income may be rewritten as

$$Y_t = \bar{G} + a(1+b)Y_{t-1} - abY_{t-2} \quad (5)$$

The recurrence relation in the income variable is a second-order linear difference equation and has a unique fixed point at

$$\bar{Y} = \frac{1}{1-a} \bar{G} \quad (6)$$

Furthermore, the conditions for stability and cyclical motion require that

$$0 < a < \frac{1}{b} < 1 \quad (7)$$

and

$$a < \frac{4b}{(1+b)^2} \quad (8)$$

respectively. Hence, if (7) and (8) hold, dampened oscillations emerge. Temporary business cycles then arise due to the interplay of the ‘multiplier’ and the ‘accelerator’: an increase in (autonomous) governmental expenditure leads to an increase in national income (via the multiplier effect) which, in turn, raises investment (via the accelerator process). Depending on the parameters  $a$  and  $b$ , national income may further increase or start to decrease. After a transient period, the ebb and flow of

economic activity vanishes and national income is equal to its long-run equilibrium value.

## III. A Simple Reformulation

Samuelson’s model has been criticized since it is not able to produce sustained fluctuations in economic activity (except for a non-generic boundary case). Moreover, empirically observed values for parameters  $a$  and  $b$  imply that the evolution of output is unstable; it runs to either plus or minus infinity. Our aim is to reformulate the model such that we obtain lasting endogenous business cycles for realistic values of  $a$  and  $b$ . In addition, we seek to emphasize the role of (heuristic) expectation formation for business cycles.

As revealed by many empirical studies, economic agents are in fact boundedly rational. According to Simon (1955), people lack, for instance, the knowledge and the computational power to derive fully optimal actions. However, this does not imply that they are completely irrational. Clearly, people strive to do the right thing. Kahneman *et al.* (1986) report that economic agents tend to use simple heuristics which have proven to be useful in the past. This observation is in line with survey evidence which indicates that agents typically use a mix of extrapolative and regressive expectation formation rules to forecast economic variables (Ito, 1990; Takagi, 1991). Similar results are observed in asset pricing experiments (Smith, 1991; Sonnemans *et al.*, 2004).

Samuelson argues that induced investment in period  $t$  is proportional to the change in national income between period  $t-1$  and  $t-2$ . Due to this lag structure, Samuelson’s model is of second-order and thus may generate temporary business cycles. We modify his investment function by postulating that induced investment is given as

$$I_t = b(E[Y_t] - Y_{t-1}) \quad (9)$$

i.e. induced investment depends on the difference between expected national income in period  $t$  and realized national income in period  $t-1$ , where  $b > 0$ .

How do agents form expectations within our model? In harmony with the empirical evidence, we assume that the expected output  $E[Y]$  may be expressed as a weighted average of an extrapolative and a regressive expectation formation rule

$$E[Y_t] = W_t E^e[Y_t] + (1 - W_t) E^r[Y_t] \quad (10)$$

According to the extrapolative predictor, agents either predict a boom or expect a recession

$$E^e[Y_t] = Y_{t-1} + c(Y_{t-1} - \bar{Y}) \quad (11)$$

where  $c$  denotes a positive extrapolation parameter and  $\bar{Y} = \bar{G}/(1-a)$  stands for the long-run equilibrium output. If output is above (below)  $\bar{Y}$ , people are optimistic (pessimistic) and predict that output will remain high (low). Regressive expectations are formalized as

$$E^r[Y_t] = Y_{t-1} + d(\bar{Y} - Y_{t-1}) \quad (12)$$

The mean-reversion parameter  $0 < d < 1$  captures the agents' expected adjustment speed of the output towards its long-run equilibrium value.

The time-varying weights associated to the two forecasting rules are subject to economic conditions. To be precise, the more the economy deviates from  $\bar{Y}$ , the less weight the agents put on the extrapolative rule. Clearly, the agents believe that extreme economic conditions are not sustainable. The relative impact of the extrapolative rule may be written as

$$W_t = \frac{1}{1 + (Y_{t-1} - \bar{Y})^2} \quad (13)$$

and thus the relative impact of the regressive rule is  $(1 - W_t)$ .

Combining (1), (2) and (9)–(13), we obtain the recurrence relation in the national income variable

$$Y_t = \bar{G} + aY_{t-1} + \frac{bd(\bar{Y} - Y_{t-1})^3 - cd(\bar{Y} - Y_{t-1})}{1 + (\bar{Y} - Y_{t-1})^2} \quad (14)$$

which is a one-dimensional nonlinear difference equation.<sup>1</sup>

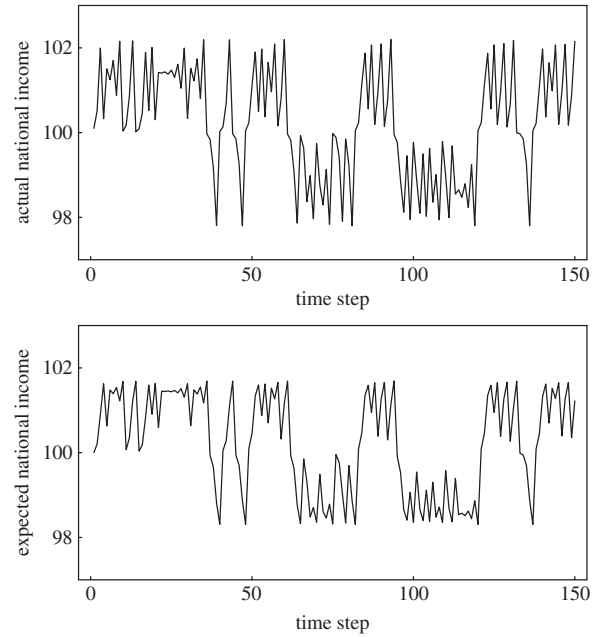
#### IV. Some Numerical Results

Let us try to understand the working of our model. Since Equation 14 is nonlinear, we continue with numerical analysis. Although simulation methods are sometimes regarded with some scepticism, we would like to stress that it is quite simple to replicate the dynamics of our model. For the simulation analysis, we fix the parameters of the model as follows:

$$\bar{G} = 0.9, \quad a = 0.9, \quad b = 4, \quad c = 1.075 \quad \text{and} \quad d = 0.5.$$

Note that a marginal propensity to consume of 0.9 and a capital to output ratio of 4 are quite realistic.

<sup>1</sup>Note that  $\bar{Y} = \bar{G}/(1-a)$  may in fact be regarded as the fundamental fixed point of Equation 14. Moreover, this fixed point is identical to the one of Samuelson's original setup.



**Fig. 1. The emergence of endogenous business cycles.**

The first (second) panel depicts the evolution of national income (expected national income) for 150 time steps. Parameter setting as in Section IV.

The parameter setting further implies that long-run equilibrium output is  $\bar{Y} = 100$ .

The top panel of Fig. 1 shows the development of national income in the time domain for 150 observations. Fluctuations in economic activity are quite intricate. For instance, there is almost no regularity in the timing and duration of booms and recessions. National income may remain above its long-run equilibrium value  $\bar{Y} = 100$  for some time, but then a sudden downturn may set in. After some time of economic distress, however, output may turn back to a higher value. Besides irregularly occurring up- and downturns, we also observe erratic movements within a boom or recession regime.

From an economic point of view there is no reason why business cycles have to occur at all. In principle, an economy can operate at its full employment level. However, business cycles do in fact occur. Within our set-up, a single exogenous shock suffices to trigger lasting endogenous swings in economic activity. What is responsible for the complex (chaotic) dynamics is the nonlinear expectation formation process of the investors. However, their expectations should not be called irrational. As confirmed by the second panel of Fig. 1, output tends to be high when investors expect it to be high. During a boom, they correctly predict that output will be above its long-run equilibrium

value. In the presence of pessimistic investors, however, the economy indeed performs badly.

What causes a regime switching from a boom to a recession and vice versa? Suppose first that the economy is in a prosperous state. As long as output is not too high, investors optimistically favour extrapolative expectations and thus induced investment tends to be positive. Since the agents also consume strongly, it is likely that output will remain elevated. However, output varies to some degree. If consumption is on a moderate level and if simultaneously induced investment turns strongly negative (investors form regressive expectations and predict a decrease in national income), output may be pushed below its long-run equilibrium value. Should that be the case, then the agents consume less and investors become depressed. As a result, the economy will linger in a (temporary) recession.

## V. Conclusions

Modern industrial economies experience significant welfare decreasing swings in economic activity. Every recession in which workers become involuntarily unemployed results in a loss of output that cannot be regained. The origins of business cycles therefore belong to the most challenging issues of macroeconomics. Several interesting mechanisms have been suggested so far which may explain business cycles (for surveys see, e.g., Medio, 1992; Day and Chen, 1993; Day, 1999; Rosser, 2000). Our focus is on the role of expectations for output variability. Guided by empirical studies, investors are assumed to use a combination of extrapolative and regressive expectation formation rules. Since regressive expectations gain in prominence if the distance between national income and its long-run equilibrium value increases, the recurrence relation that determines national income is nonlinear. Our main finding is that the mood of the investors may stimulate complex output fluctuations. If investors are optimistic (pessimistic) output tends to be above (below) its long-run equilibrium value. This observation is important for policy makers. If they succeed in creating an

optimistic atmosphere, a recession may be shortened. Moreover, our model produces sustained and intricate fluctuations in economic activity for realistic values of the marginal propensity to consume and the capital to output ratio.

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