‘...THE PARTNER OF 4 IS PLUS 10 OF THIS PARTNER’ – YOUNG CHILDREN MAKE SENSE OF TASKS ON FUNCTIONAL RELATIONS

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In the German perspective early algebra and –more important– the first steps towards the development of algebraic thinking in elementary mathematics are often neglected. This article focuses on the interpretation of ‘potentially algebraic’ tasks by primary school children. The different ways they make sense of the tasks will give information about the important phase of transition, where arithmetical routines fail and new strategies have to come to life.

Some former German studies (cf. Bardy, 1993; Meißner, 1987; Vollrath, 1983, 1989) on functional understanding in early years have focused on the abilities of the participating children to solve the presented tasks. The definition of ‘functional thinking’ was understood differently. On the one hand ‘functional thinking’ was interpreted as thinking relationally, i.e. seeing some relations without the need to recognise the function as an object. On the other hand in some studies ‘functional thinking’ was used synonymous to ‘thinking with function’ and being aware of the function as an object. ‘Encapsulation’ (Davis, et al., 1997) of function as an object is no subject for primary school children. One neither expects nor wants to make the children construing an object of function as such. This marks one of the reasons why the results of these studies were not taken into account for further developments of elementary school mathematics in Germany. In fact German research in this field still focuses very much on higher grades. Functional relations and other themes related to pre- or early algebra are barely mentioned in the primary school syllabus.

This fact can be either seen as an advantage or a disadvantage. There is no data proving the results of foreign researcher (cf. e.g. Warren, 2001) for German children. But it is a chance as well to be the first confronting children with tasks on early algebra. At least a kind of spontaneous reaction can be observed, because the influence by school mathematics is negligible.

The focus of the presented study is on *the very first beginning* of a process leading to thinking about mathematical functions symbolically. Regarding to Tall’s, et al., (2000) sophistication of development, this research is located between ‘routine mathematics’ and ‘performing mathematics flexible’. Symbols and forms of representation (cf. Carraher, 2001) are not the key interest here but shifts in interpretation of the task by the children (cf. Yackel, 1997). Leaving behind the routine forms of solving tasks demands a qualitative shift in reading and understanding the task. One possibility to describe this shift is to associate the routine procedures with numerical, arithmetical ways and the more flexible ways with pre-algebraic ones. It is of special interest here if a supposed ‘transition phase’ can be detected in the analysis of children’s interaction with tasks on functional relations, i.e. interaction while solving the tasks in a individual way or discussing the solution. This leads to two basic research questions: *Do* young children handle tasks on functional relations? and *How* do they make sense of it (within their very own framing)?

Methodology and Sample

One part of recent studies on primary school children's understanding of number patterns (Steinweg, 2001a, 2001b) dealt with functional relations. One special task, which can formally be noted as $y = 7x + 1$, was a subject amongst others in the three different approaches, i.e. a written test, interviews, and a school-project.

number	0	2	3	6	7	10	
partner	1	15	22	43		71	78

Figure 1

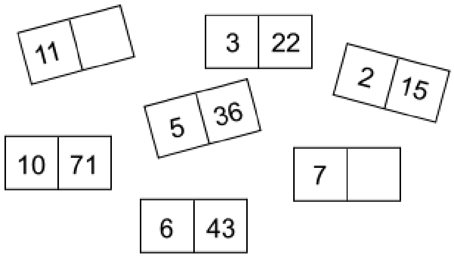

Figure 2

Within a paper-pencil-test the functional relation was presented in a table (fig. 1) showing 5 pairs of related values of y and x. The children were asked to complete the table by filling in one missing y (for $x = 7$) and one missing x (for $y = 78$). Letters as variables were not used. The independent variable was labelled as 'number', the dependent one as 'partner'.

The interviews used a different presentation by noting each pair on some kind of card (fig. 2). The cards were in no order, which means this representation indicates no numerical order for the x-values. The children were expected to accept 'missing x's in the number-line' more easily. The interviewees were asked to find the partners (related values y) for the 'lonely' numbers 7 and 11.

Two classes of one school took part in a school-project about patterns. The first time these children were confronted with functional relations, each pair of number and partner number were noted on a single piece of paper. The children were allowed to order the sheets. The task on the function $y = 7x+1$ was offered to them in the written form shown in figure 2 on a working sheet during the project.

63 children (9-10-year-olds, after 4 years of formal schooling) of three different schools and different socio-ethic-backgrounds participated in the written test. 15 students (9-10-year-olds of three other schools; 5 children per school) were interviewed. 46 children (23 in each of the classes) of the same age participated in the school project.

Research Questions

Because functional relations are not mentioned in the German primary school curriculum, i.e. they are not taught in the daily school life, the first question was to find out if the children can handle the offered tasks and can make sense of the presentation form. The second –and more important– question was: *How* do they make sense of it?

Quantitative Results – Do They Handle the Tasks?

As a first step the answers given and observed in the different research approaches were categorised in four categories (Steinweg, 2001b) as shown in the table below (fig. 3).

%	Paper-Pencil-Test	Inter-views	School-Project
filled in both missing values	34,2	60,0	79,1
filled in one missing value	5,5	6,7	9,3
filled in numbers without an apparent connection with the relation	34,9	26,6	6,9
not worked on	25,4	6,7	4,7
total	100	100	100

Figure 3

The quantitative results are encouraging. The children were able to cope with the demands of the task –even though they did not have any experiences with functional relations in mathematics lessons at all. The table-representation (fig. 1) seemed to be harder to work on. The children had to find one missing dependent and one independent value whereas in the other studies they had to find two dependent ones. Within the interaction of the interviews or the school-project far more than 50% were able to find the ‘mathematically correct’ or the usual answers to the missing values.

These figures do not give any answer to the question *how* the children found access to the task and how they used the structures of the functional relation. The mathematical structure –presented in different forms– is in no way self-explicit. The children had to make up their own sense and their interpretations are embedded in their social backgrounds (cf. e.g. Voigt, 1995).

Two case-studies of quite different ways of interpretation will be given to shed light on this problem.

Qualitative Results – How Do They Make Sense of the Tasks?

The transcripts of the interviews were analysed to find out if the existence of a transition phase between arithmetical-numerical thinking and pre-algebraic thinking can be proved.

The two children presented in the case studies below both belong to the majority of ‘correct’ solvers, even though they react and interact very differently. In a way the interpretations stand for two qualitatively, completely different ways of thinking leading to the same, quantitative result.

Two Cases – Anja and Sven

Anja and Sven, two fourth-graders (10-year-olds), participated in the interviews and found both missing partners for $x = 7$ and $x = 11$ (cf. figure 2). They also found the partners for $x = 7$ and $x = 12$ in the following task on the relation $y = 10x-5$ where the pairs 2/15, 4/35, 5/45, 10/95 and 11/105 are given.

Anja and Sven interpreted the tasks in an individual way. (The following transcripts were translated by the author. The analysis is based on the German originals).

Anja did not search for an interaction with the interviewer. She solved the task silently on her own and had to be asked about her thoughts:

66	*A*	[She thinks about the task and completes the missing values silently.]
67	*I*	*how do you found it out?*
68	*A*	*because first I looked at it. the 2 and the 3, and then I calculated 15 plus what makes 22. that was 7. and then I have looked at 5 and 6 again. that was also 7. then I have calculated here with the 10 71 plus 7, and then I wrote it down as a partner for 11, and then with the 6 I have then calculated 43 plus 7 equals 50, and then I wrote it down here as a partner for 7.*

Anja's thoughts had to be reconstructed out of her only comment (68). She interpreted the task as a relation between numbers and consecutive numbers in order to find the missing values. She set up a relation between the pairs 2 and 15, and 3 and 22. She calculated the difference of the values of y (22 – 15 = 7). She proved her thought by focussing on two different pairs (5 and 36, 6 and 43). The detection of the same difference encouraged her to stick to her interpretation. She used the structures to find out the missing partner for x = 11 by adding 7 to the y-value of 10; by analogy she calculated the y-value for 7 by adding 7 to f(6) = 43.

The functional relation between x and y can be made explicit via an algebraic term or be determined by a recursive procedure. The focus is here on the relation between x and x+1. Anja interpreted the task in this way. She had to order the given pairs mentally to do so. She gave a new structure to them by doing so. Her interpretation was successful because two pairs belonging to consecutive numbers were given. Unfortunately a cognitive conflict could not be initiated by the task. This chance was missed because the representation form was not checked on this focus beforehand. The second task offered Anja the chance to stick to her interpretation.

70	*A*	[She thinks about the task and completes the missing values silently.] *and there you always have to ... hmm the partner of 4 is plus 10 of this partner. and 35 plus what makes 45? that is 10. and then I have looked up 10 and 11 again. there again is 10 more. and then I have calculated, of 45 the partner is 5 and then I have calculated plus 20 that is 65. therefore the partner of 7 is 65. with 11 ...105 I have calculated simply plus 10 and that is 115.*
71	*I*	*hmm.*

Consequently Anja interpreted the second tasks in the same effective way she used for the first one. She determined pairs belonging to consecutive numbers, which led her to the solution. Her consistent interpretation made her able to even cope with 'jumps' in the numerical order of the given x-values (70). Without any hesitation she doubles the value she had to add to a given y-value related to x = 5 to determine the missing partner of 7.

Sven interacted with the interviewer more openly than Anja. The development of his thoughts and interpretation can be reconstructed more easily:

1 *S* [whispers] *what is the partner?*
2 *I* *do you know what the task is about, hmm – what partner is searched for?*
3 *S* *hmm, always the opposite one* [points at a given pair of values]
4 *I* *ah, yes, that's it.*
5 *S* [thinks about it silently]
I can find always– funny– that there are various differences.
6 *I* *hmm.*
7 *S* *that is eventually –hmm– one moment, oh, I see, now I know, that is always times 7 and then plus 1.*
8 *I* *how do you found it out yet?*
9 *S* *because I have seen I was interested in the tables in former times, and then*
10 *I* *what?*
11 *S* *the tables, I enjoyed to do them in former times*
12 *I* *oh, I see.*
13 *S* *and here it strikes me, 36 that is 7 times 5 plus 1.*
14 *I* *ah, yes.*
15 *S* *here it has to be 11 times 7 are 77 – 78!*
16 *I* *hmm.*
17 *S* *and here 7 times 7 equals 49 – 50.*

First Sven tried to find a structure by calculating the differences (5). After his first interpretation led him into nothing he changed thought very fast and found out the relation between the x- and y-values and formulated it as a general rule (7). The suddenness of the finding is typical and could be observed in several other interviews as well.

Sven used his (school-)knowledge. He was not able to make his thoughts any more explicit but stating his former favour for the tables (9/11). Sven knew his tables and could make use of them. The tables can be regarded as one pre-structure of a functional relation. They form a relation between two numbers by a certain rule. Sven had to extend the rule by adding 'plus 1' and could easily generate a general rule for the functional relation offered here. By doing so he found a way to calculate each and every y-value asked for in an explicit way.

Like Anja Sven stuck to his interpretation of the task with a certain consistence. He explicitly suggested the second task to be of the 'same' kind (19) and indicated that he was going to keep his way of thinking.

19 *S* *this might be the same with other numbers ...*
20 *I* *hmm.*
21 *S* [thinks about it first silently then whispering]
22 *I* *What numbers are you checking on?*

23 S *nothing. that is – hmm – searching half of the tables which one might suite ... I think I got something* [his eyes shifted to and fro between two number-pairs ...] *yes, that's right. it's the 10th-table minus 5.*

24 I *hmm. wow!*

25 S *that has to be 115.*

26 I *hmm.*

27 S *and this one 65.*

Sven admitted to search the tables without a certain idea at the beginning (23). One has to take into account that it was neither possible for Sven to know the fitting table nor the additive aspect of the functional relation at this point; although his trial-and-error procedure was very effective. This was due to his good pre-knowledge about the tables. One has to be aware of the fact that his thoughts are guided by the given examples –his interpretation was based on empirical facts always. Sven's 'checking' of his hypothesis by looking at a second example (23) indicated the importance of empirical data.

The two different interpretations of Anja and Sven can be summarised in the figures below:

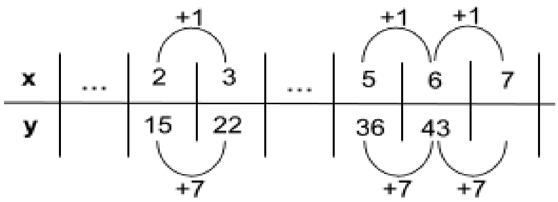

Figure 4

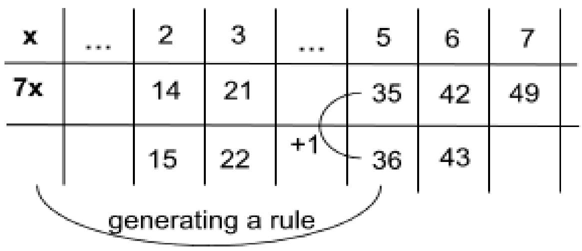

Figure 5

Anja followed a recursive way. Because the given pairs represent the relation between consecutive numbers for x, Anja was able to find the missing y-values. The functional relation between a particular pair of x and y was not important in her interpretation but the relation between a given x and its y-value to (x+1) and its y-value (fig. 4). If x → a, then x+1 → a+7.

Sven compared the given pairs with the 'function' y = 7x (the tables) and was able to give the function y = 7x+1 in his own words as a general rule (fig. 5).

Discussion

Algebraic solutions and interpretations do not have to include the use of letters. One might solve a task algebraically without even thinking or knowing of letters as variables. Schliemann, et al., (2001) describe the working on particular numbers and computing numerical answers in contrast to working on sets of numbers and describing relations among variables. Anja and Sven stuck to the given examples and did not work on sets. Solving similar tasks they both used consistent interpretations. In fact they both did more than just guess-and-check on numerical basis. They focused on their interpreted relation, a structure, and did not guess any value for the missing partners. Their arithmetical routines were left behind. Suitable solutions could only be found via structural ways.

Mathematical objects, like the functional relation described by $y = 7x+1$, as artefacts can be interpreted in different ways and with various depth. The interpretation once used for a solution can change while solving the same problem and varies from one individual to another. The task itself has to offer the chance to view the relation as a function algebraically. In order to support the students the task has to ask for indirect, systematic, or reverse strategies (cf. Steinbring, 1999). The representation has to be understood as such, and has to lead to the underlying mathematical structure.

The results give rise to a new research question: Is it possible to guide the students to see the functional relation as such? Additional questions could lead in this way. It will be important to ask for dependent variables beyond single-digit values (What partner belongs to 100?) to illustrate the continuity of the relation. One has to ask for independent variables as well to support flexible reverse thinking (What number belongs to the partner 36?) The learner should become aware of the underlying rule (What is the rule?) and last but not least the graphical representation has to be added (What does the (function)graph look like?) (cf. e.g. Douady, 1999). At this point further questions could guide the children to focus on the graph as one possible representation for the function as an artificial mathematical object. The children should get to know other graphs in different situation to point out the special quality of a linear function and its graph. These problems will be worked on in the course of this study. Functions are far more than pairs of depending values (cf. e.g. Führer, 1985).

The children introduced here were not using variables but their uttered thoughts indicate some mathematical activities, which do no longer belong to routine mathematics –for them. This shows that even in elementary mathematics ‘unspoken changes of perspectives’ (Malara & Iaderose, 1999) could be identified. These changes have to be made ‘speakable’ for the children and for the teachers. An awareness for the transitions, which are founded very early and are regarded by Dooren, et al., (2001) as ‘precursors for algebraic thinking’ (p. 359), is vital. The shown examples put the finger on the important pivotal where the change of perspective can be successfully made. This might be the only way to prevent the constitution of two different ‘mathematics’ as one can find in the reactions of secondary school children (Cerulli & Mariotti, 2001).

The wide held opinion that the main goal to be achieved in elementary mathematics is mastering of arithmetical routines has to be reconsidered. The learner should become aware of the limits of the routines. More emphasis has to be put on the flexible use of mathematics and on fostering the individual strategies, which pave the way for structural ways of thinking. The design of suitable, ‘potentially algebraic’ tasks that will enrich primary school mathematics will be the major challenge of further research.

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